

Eight state supersymmetric U model of strongly correlated fermions

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An integrable eight state supersymmetric U model is proposed, which is a fermion model with correlated single-particle and pair hoppings as well as uncorrelated triple-particle hopping. It has an $gl(3|1)$ supersymmetry and contains one symmetry-preserving free parameter. The model is solved and the Bethe ansatz equations are obtained.

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Exactly solvable models of strongly correlated fermions have in recent years generated a great deal of attention, since they are believed to play a promising role in unraveling the mystery of high- T_c superconductivity (see, e.g. [1]). Several integrable correlated fermion systems have so far appeared in the literature. Most famous are the supersymmetric t - J model and the Hubbard model. Other integrable correlated electron systems of interest include the extended Hubbard model [2], the supersymmetric U model proposed in [3] and extensively investigated in [4–6], and its q -deformed version [7,8], the model proposed in [9] and its generalization [10].

In this communication, we propose an eight state version of the supersymmetric U model. It is a supersymmetric fermion model with correlated single particle and pair hoppings as well as uncorrelated triple-particle hopping. We then solve the model by means of coordinate Bethe ansatz method and the Bethe ansatz equations are derived.

Let $c_{j,\alpha}^\dagger$ ($c_{j,\alpha}$) denotes a fermionic creation (annihilation) operator which creates (annihilates) a fermion of species $\alpha = +, 0, -$ at site j . These operators satisfy the anti-commutation relations given by $\{c_{i,\alpha}^\dagger, c_{j,\beta}\} = \delta_{ij}\delta_{\alpha\beta}$, where $i, j = 1, 2, \dots, L$ and $\alpha, \beta = +, 0, -$. At a given lattice site j there are eight possible states:

$$\begin{aligned} &|0\rangle, \quad c_{j,+}^\dagger|0\rangle, \quad c_{j,0}^\dagger|0\rangle, \quad c_{j,-}^\dagger|0\rangle, \\ &c_{j,+}^\dagger c_{j,0}^\dagger|0\rangle, \quad c_{j,+}^\dagger c_{j,-}^\dagger|0\rangle, \quad c_{j,0}^\dagger c_{j,-}^\dagger|0\rangle, \quad c_{j,+}^\dagger c_{j,0}^\dagger c_{j,-}^\dagger|0\rangle. \end{aligned} \quad (1)$$

By $n_{j,\alpha} = c_{j,\alpha}^\dagger c_{j,\alpha}$ we denote the number operator for the fermion of species α at site j .

In the sequel we only consider periodic lattice of length L . The Hamiltonian for our new model reads

$$\begin{aligned} H(g) &= \sum_{j=1}^L H_{j,j+1}(g), \\ H_{j,j+1}(g) &= - \sum_{\alpha} (c_{j,\alpha}^\dagger c_{j+1,\alpha} + \text{h.c.}) \exp \left\{ -\frac{\eta}{2} \sum_{\beta(\neq\alpha)} (n_{j,\beta} + n_{j+1,\beta}) + \frac{\zeta}{2} \sum_{\beta\neq\gamma(\neq\alpha)} (n_{j,\beta} n_{j,\gamma} + n_{j+1,\beta} n_{j+1,\gamma}) \right\} \\ &\quad - \frac{1}{2(g+1)} \sum_{\alpha\neq\beta\neq\gamma} (c_{j,\alpha}^\dagger c_{j,\beta}^\dagger c_{j+1,\beta} c_{j+1,\alpha} + \text{h.c.}) \exp \left\{ -\frac{\xi}{2} (n_{j,\gamma} + n_{j+1,\gamma}) \right\} \\ &\quad - \frac{2}{(g+1)(g+2)} (c_{j,+}^\dagger c_{j,0}^\dagger c_{j,-}^\dagger c_{j+1,-} c_{j+1,0} c_{j+1,+} + \text{h.c.}) \\ &\quad + \sum_{\alpha} (n_{j,\alpha} + n_{j+1,\alpha}) - \frac{1}{2(g+1)} \sum_{\alpha\neq\beta} (n_{j,\alpha} n_{j,\beta} + n_{j+1,\alpha} n_{j+1,\beta}) \\ &\quad + \frac{2}{(g+1)(g+2)} (n_{j,+} n_{j,0} n_{j,-} + n_{j+1,+} n_{j+1,0} n_{j+1,-}), \end{aligned} \quad (2)$$

where

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$$\eta = -\ln \frac{g}{g+1}, \quad \zeta = \ln(g+1) - \frac{1}{2} \ln g(g+2), \quad \xi = -\ln \frac{g}{g+2}. \quad (3)$$

As will be seen, the supersymmetry algebra underlying this model is $gl(3|1)$. Remarkably, the model still contains the parameter g as a free parameter without breaking the supersymmetry. Also this model is exactly solvable on the one dimensional periodic lattice, as is seen below.

Our local hamiltonian $H_{i,j}(g)$ does not act as graded permutation of the states (1) at sites i and j . If one projects out any one species, then one singlet, two doublets and one triplet are projected out and the remaining four states are: one hole, two singlets and one doublet. In this case, the projected hamiltonian is nothing but that of the supersymmetric U model [3] with the parameter U (in the supersymmetric U model) related to the parameter g here via $U = \pm \frac{2}{g+1}$. It is in this sense that the present model is an eight state version of the supersymmetric U model.

To show that (2) is $gl(3|1)$ supersymmetric, we denote the generators of $gl(3|1)$ by E_ν^μ , $\mu, \nu = 1, 2, 3, 4$ with grading $[1] = [2] = [3] = 0$, $[4] = 1$. In a typical 8-dimensional representation of $gl(3|1)$, the highest weight $\Lambda = (0, 0, 0|g)$ itself of the representation depends on the free parameter g , thus giving rise to a one-parameter family of inequivalent irreps. Let $\{|x\rangle\}_{x=1}^8$ denote an orthonormal basis with $|1\rangle, |5\rangle, |6\rangle, |7\rangle$ even (bosonic) and $|2\rangle, |3\rangle, |4\rangle, |8\rangle$ odd (fermionic). Then the simple generators $\{E_\mu^\mu\}_{\mu=1}^4$ and $\{E_{\mu+1}^\mu, E_{\mu+1}^{\mu+1}\}_{\mu=1}^3$ are 8×8 supermatrices of the form

$$\begin{aligned} E_2^1 &= |3\rangle\langle 4| + |5\rangle\langle 6|, & E_1^2 &= |4\rangle\langle 3| + |6\rangle\langle 5|, & E_1^1 &= -|4\rangle\langle 4| - |6\rangle\langle 6| - |7\rangle\langle 7| - |8\rangle\langle 8|, \\ E_3^2 &= |2\rangle\langle 3| + |6\rangle\langle 7|, & E_2^3 &= |3\rangle\langle 2| + |7\rangle\langle 6|, & E_2^2 &= -|3\rangle\langle 3| - |5\rangle\langle 5| - |7\rangle\langle 7| - |8\rangle\langle 8|, \\ E_4^3 &= \sqrt{g}|1\rangle\langle 2| + \sqrt{g+1}(|3\rangle\langle 5| + |4\rangle\langle 6|) + \sqrt{g+2}|7\rangle\langle 8|, \\ E_3^4 &= \sqrt{g}|2\rangle\langle 1| + \sqrt{g+1}(|5\rangle\langle 3| + |6\rangle\langle 4|) + \sqrt{g+2}|8\rangle\langle 7|, \\ E_3^3 &= -|2\rangle\langle 2| - |5\rangle\langle 5| - |6\rangle\langle 6| - |8\rangle\langle 8|, \\ E_4^4 &= g|1\rangle\langle 1| + (g+1)(|2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|) + (g+2)(|5\rangle\langle 5| + |6\rangle\langle 6| + |7\rangle\langle 7|) + (g+3)|8\rangle\langle 8|. \end{aligned} \quad (4)$$

The non-simple generators are obtained from the simple ones by using the defining (anti-)commutation relations of $gl(3|1)$, which we omit. Further choose

$$\begin{aligned} |1\rangle &= |0\rangle, & |2\rangle &= c_{j,+}^\dagger |0\rangle, & |3\rangle &= c_{j,0}^\dagger |0\rangle, & |4\rangle &= c_{j,-}^\dagger |0\rangle, \\ |5\rangle &= c_{j,+}^\dagger c_{j,0}^\dagger |0\rangle, & |6\rangle &= c_{j,+}^\dagger c_{j,-}^\dagger |0\rangle, & |7\rangle &= c_{j,0}^\dagger c_{j,-}^\dagger |0\rangle, & |8\rangle &= c_{j,+}^\dagger c_{j,0}^\dagger c_{j,-}^\dagger |0\rangle. \end{aligned} \quad (5)$$

Then the verification that the hamiltonian $H(g)$ commutes with all generators of $gl(3|1)$ is just a straightforward calculation.

The model is exactly solvable by the Bethe ansatz, since the local hamiltonian $H_{i,i+1}(g)$ is actually derived from a $gl(3|1)$ -invariant rational R-matrix [which satisfies the graded Yang-Baxter equation]. The R-matrix is given by [11]

$$\check{R}(u) = \check{P}_1 - \frac{u+2g}{u-2g} \check{P}_2 + \frac{(u+2g)(u+2g+2)}{(u-2g)(u-2g-2)} \check{P}_3 - \frac{(u+2g)(u+2g+2)(u+2g+4)}{(u-2g)(u-2g-2)(u-2g-4)} \check{P}_4, \quad (6)$$

where \check{P}_k , $k = 1, 2, 3, 4$, are four projection operators:

$$\check{P}_1 = \sum_{k=1}^8 |\Psi_k^1\rangle\langle \Psi_k^1|, \quad \check{P}_4 = \sum_{k=1}^8 |\Psi_k^4\rangle\langle \Psi_k^4|, \quad \check{P}_2 = \frac{1}{2}(I + P) - \check{P}_1, \quad \check{P}_3 = \frac{1}{2}(I - P) - \check{P}_4, \quad (7)$$

where P is the graded permutation operator and $|\Psi_k^1\rangle$, $|\Psi_k^4\rangle$, $k = 1, 2, \dots, 8$ are given by

$$\begin{aligned} |\Psi_1^1\rangle &= |1\rangle \otimes |1\rangle, & |\Psi_2^1\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \otimes |1\rangle + |1\rangle \otimes |2\rangle), \\ |\Psi_3^1\rangle &= \frac{1}{\sqrt{2}}(|3\rangle \otimes |1\rangle + |1\rangle \otimes |3\rangle), & |\Psi_4^1\rangle &= \frac{1}{\sqrt{2}}(|4\rangle \otimes |1\rangle + |1\rangle \otimes |4\rangle), \\ |\Psi_5^1\rangle &= \frac{1}{\sqrt{2(2g+1)}}[\sqrt{g+1}(|5\rangle \otimes |1\rangle + |1\rangle \otimes |5\rangle) + \sqrt{g}(|2\rangle \otimes |3\rangle - |3\rangle \otimes |2\rangle)], \\ |\Psi_6^1\rangle &= \frac{1}{\sqrt{2(2g+1)}}[\sqrt{g+1}(|6\rangle \otimes |1\rangle + |1\rangle \otimes |6\rangle) + \sqrt{g}(|2\rangle \otimes |4\rangle - |4\rangle \otimes |2\rangle)], \\ |\Psi_7^1\rangle &= \frac{1}{\sqrt{2(2g+1)}}[\sqrt{g+1}(|7\rangle \otimes |1\rangle + |1\rangle \otimes |7\rangle) + \sqrt{g}(|3\rangle \otimes |4\rangle - |4\rangle \otimes |3\rangle)], \end{aligned}$$

$$\begin{aligned}
|\Psi_8^1\rangle &= \frac{1}{2\sqrt{2g+1}}[\sqrt{g}(|2\rangle \otimes |7\rangle + |7\rangle \otimes |2\rangle + |5\rangle \otimes |4\rangle + |4\rangle \otimes |5\rangle - |3\rangle \otimes |6\rangle - |6\rangle \otimes |3\rangle) + \sqrt{g+2}(|8\rangle \otimes |1\rangle + |1\rangle \otimes |8\rangle)], \\
|\Psi_1^4\rangle &= \frac{1}{2\sqrt{2g+3}}[\sqrt{g+2}(|7\rangle \otimes |2\rangle - |2\rangle \otimes |7\rangle + |5\rangle \otimes |4\rangle - |4\rangle \otimes |5\rangle - |6\rangle \otimes |3\rangle + |3\rangle \otimes |6\rangle) + \sqrt{g}(|1\rangle \otimes |8\rangle - |8\rangle \otimes |1\rangle)], \\
|\Psi_2^4\rangle &= \frac{1}{\sqrt{2(2g+3)}}[\sqrt{g+1}(|8\rangle \otimes |2\rangle + |2\rangle \otimes |8\rangle) + \sqrt{g+2}(|5\rangle \otimes |6\rangle - |6\rangle \otimes |5\rangle)], \\
|\Psi_3^4\rangle &= \frac{1}{\sqrt{2(2g+3)}}[\sqrt{g+1}(|8\rangle \otimes |3\rangle + |3\rangle \otimes |8\rangle) + \sqrt{g+2}(|5\rangle \otimes |7\rangle - |7\rangle \otimes |5\rangle)], \\
|\Psi_4^4\rangle &= \frac{1}{\sqrt{2(2g+3)}}[\sqrt{g+1}(|8\rangle \otimes |4\rangle + |4\rangle \otimes |8\rangle) + \sqrt{g+2}(|6\rangle \otimes |7\rangle - |7\rangle \otimes |6\rangle)], \\
|\Psi_5^4\rangle &= \frac{1}{\sqrt{2}}(-|8\rangle \otimes |5\rangle + |5\rangle \otimes |8\rangle), \quad |\Psi_6^4\rangle = \frac{1}{\sqrt{2}}(-|8\rangle \otimes |6\rangle + |6\rangle \otimes |8\rangle), \\
|\Psi_7^4\rangle &= \frac{1}{\sqrt{2}}(-|8\rangle \otimes |7\rangle + |7\rangle \otimes |8\rangle), \quad |\Psi_8^4\rangle = |8\rangle \otimes |8\rangle,
\end{aligned} \tag{8}$$

which are easily seen to be orthonormal, so that

$$\begin{aligned}
\langle \Psi_k^1 | &= (|\Psi_k^1\rangle)^\dagger, \quad \langle \Psi_k^4 | = (|\Psi_k^4\rangle)^\dagger, \quad k = 1, \dots, 8, \\
(|x\rangle \otimes |y\rangle)^\dagger &= (-1)^{[|x|][|y|]} |y\rangle \langle x|.
\end{aligned} \tag{9}$$

Here $[|x|] = 0$ for even (bosonic) $|x\rangle$ and $[|x|] = 1$ for odd (fermionic) $|x\rangle$.

On the L -fold tensor product space we denote $\check{R}(u)_{j,j+1} = I^{\otimes(j-1)} \otimes \check{R}(u) \otimes I^{\otimes(L-j-1)}$, and define the local hamiltonian by

$$H_{j,j+1}^R(g) = \left. \frac{d}{du} \check{R}_{j,j+1}(u) \right|_{u=0} = -4(2g+1)(\check{P}_1)_{j,j+1} + \frac{4g(2g+3)}{g+2}(\check{P}_4)_{j,j+1} + 2g P_{j,j+1}. \tag{10}$$

Then by (7), (8), (9) and (5), and after tedious but straightforward manipulation, one gets, up to a constant, $H_{j,j+1}(g) = \frac{1}{2(g+1)} H_{j,j+1}^R(g)$. [This identity also indicates that $H(g)$ commutes with all generators of $gl(3|1)$, since the rational R-matrix $\check{R}(u)$ is a $gl(3|1)$ invariant.]

As mentioned above, the system is exactly solvable by means of Bethe ansatz technique. We assume the following wavefunction

$$\psi_{\alpha_1, \dots, \alpha_N}(x_1, \dots, x_N) = \sum_P \epsilon_P A_{\alpha_{Q_1}, \dots, \alpha_{Q_N}}(k_{P_{Q_1}}, \dots, k_{P_{Q_N}}) \exp\left(i \sum_{j=1}^N k_{P_j} x_j\right), \tag{11}$$

where Q is the permutation of the N particles such that $1 \leq x_{Q_1} \leq \dots \leq x_{Q_N} \leq L$. Denote $X_Q = \{x_{Q_1} \leq \dots \leq x_{Q_N}\}$. The coefficients $A_{\alpha_{Q_1}, \dots, \alpha_{Q_N}}(k_{P_{Q_1}}, \dots, k_{P_{Q_N}})$ from regions other than X_Q are connected with each other by elements of two-particle S-matrix:

$$S_{1,2}(k_1, k_2) = \frac{\theta(k_1) - \theta(k_2) + ic\mathcal{P}_{12}}{\theta(k_1) - \theta(k_2) + ic}, \tag{12}$$

where operator \mathcal{P}_{12} interchanges the species variables α_1 and α_2 ($\alpha_1, \alpha_2 = +, 0, -$), the rapidities $\theta(k_j)$ are related to the single-particle quasi-momenta k_j by $\theta(k) = \frac{1}{2} \tan(\frac{k}{2})$ and the dependence on the system parameter g is incorporated in the parameter $c = e^\eta - 1 = 1/g$. The periodicity condition for the system on the finite interval $(0, L)$ results in the Bethe equations for the set of N momenta $k_j : \exp(ik_j L) = T_j$, $j = 1, \dots, N$, where

$$T_j = S_{j,j+1}(k_j, k_{j+1}) \cdots S_{j,N}(k_j, k_N) S_{j,1}(k_j, k_1) \cdots S_{j,j-1}(k_j, k_{j-1}), \quad j = 1, \dots, N. \tag{13}$$

The meaning of T_j is the scattering matrix of the j -th particle on the other $(N-1)$ particles. So now the problem is to diagonalize T_j to arrive at a system of scalar equations. It is easy to show that $T_j = \tau(\lambda = k_j)$, where

$$\tau(\lambda) = \text{tr}_0 [S_{0,1}(\lambda - k_1) \cdots S_{0,N}(\lambda - k_N)] \tag{14}$$

is the transfer matrix of the inhomogeneous $gl(3)$ -spin magnet of N sites. The commutativity of the transfer matrix for different values of the spectral parameter λ implies that T_j , $j = 1, \dots, N$ can be diagonalized simultaneously. The Bethe ansatz equations are written in terms of the rapidities $\theta_j \equiv \theta(k_j)$ and λ_σ

$$\begin{aligned}
e^{ik_j L} &= \prod_{\sigma=1}^{M_1} \frac{\theta_j - \lambda_\sigma^{(1)} + ic/2}{\theta_j - \lambda_\sigma^{(1)} - ic/2}, \\
\prod_{j=1}^N \frac{\lambda_\sigma^{(1)} - \theta_j + ic/2}{\lambda_\sigma^{(1)} - \theta_j - ic/2} &= - \prod_{\rho=1}^{M_1} \frac{\lambda_\sigma^{(1)} - \lambda_\rho^{(1)} + ic}{\lambda_\sigma^{(1)} - \lambda_\rho^{(1)} - ic} \prod_{\rho=1}^{M_2} \frac{\lambda_\sigma^{(1)} - \lambda_\rho^{(2)} - ic/2}{\lambda_\sigma^{(1)} - \lambda_\rho^{(2)} + ic/2}, \quad \sigma = 1, \dots, M_1, \\
\prod_{\rho=1}^{M_1} \frac{\lambda_\gamma^{(2)} - \lambda_\rho^{(1)} + ic/2}{\lambda_\gamma^{(2)} - \lambda_\rho^{(1)} - ic/2} &= - \prod_{\rho=1}^{M_2} \frac{\lambda_\gamma^{(2)} - \lambda_\rho^{(2)} + ic}{\lambda_\gamma^{(2)} - \lambda_\rho^{(2)} - ic}, \quad \gamma = 1, \dots, M_2,
\end{aligned} \tag{15}$$

The energy of the system in the state corresponding to the sets of solutions $\{\theta_j\}$ and $\{\lambda_\sigma\}$ is (up to an additive constant, which we drop) $E = -2 \sum_{j=1}^N \cos k_j$.

To summarize, we have presented an integrable eight state version of the supersymmetric U model. It is a $gl(3|1)$ supersymmetric fermion model with generalized hoppings. We have solved the model by the coordinate Bethe ansatz method and derived the Bethe ansatz equations. There are many things remained to be done for this new model. One of them is to study physical properties, such as the phase diagram and the critical exponents, of the system. It is also interesting to incorporate integrable boundary conditions into the model and to investigate finite-size corrections of the boundary system. We hope to report results on those aspects in future publications.

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